The square of white noise as a Jacobi field

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Abstract

We identify the representation of the square of white noise obtained by L. Accardi, U. Franz and M. Skeide in [*Comm. Math. Phys.* **228** (2002), 123–150] with the Jacobi field of a Lévy process of Meixner's type.

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1 Formulation of the result

The problem of developing a stochastic calculus for higher powers of white noise, i.e., "nonlinear stochastic calculus", was first stated by Accardi, Lu, and Volovich in [4]. Since the white noise is an operator-valued distribution, in order to solve this problem one needs an appropriate renormalization procedure. In [5, 6], it was proposed to renormalize the commutation relations and then to look for Hilbert space representations of them. Let us shortly discuss this approach.

We will use $\mathbb{R}^d$, $d \in \mathbb{N}$, as an underlying space. Let $b(x)$, $x \in \mathbb{R}^d$, be an operator-valued distribution satisfying the canonical commutation relations:

$$\begin{aligned} [b(x), b(y)] = [b^\dagger(x), b^\dagger(y)] = \mathbf{0}, \\ [b(x), b^\dagger(y)] = \delta(x-y)\mathbf{1}. \end{aligned} \tag{1}$$

Here, $[A, B]{:=}AB - BA$ and $b^\dagger(x)$ is the dual operator of $b(x)$. Denote

$$B_x{:=}b(x)^2, \quad B_x^\dagger{:=}b^\dagger(x)^2, \quad N_x{:=}b^\dagger(x)b(x), \quad x \in \mathbb{R}^d. \tag{2}$$

One wishes to derive from (1) the commutation relations satisfied by the operators $B_x, B_x^\dagger, N_x$. To this end, one needs to make sense of the square of the delta function, $\delta(x)^2$. But it is known from the distribution theory that

$$\delta(x)^2 = c\delta(x), \tag{3}$$

where $c \in \mathbb{C}$ is an arbitrary constant (see [5] for a justification of this formula and bibliographical references).

Thus, using (1) and formula (3) as a renormalization, we get

$$\begin{aligned} [B_x, B_y^\dagger] &= 2c\delta(x-y)\mathbf{1} + 4\delta(x-y)N_y, \\ [N_x, B_y^\dagger] &= 2\delta(x-y)B_y^\dagger, \\ [N_x, B_y] &= -2\delta(x-y)B_y, \\ [N_x, N_y] = [B_x, B_y] &= [B_x^\dagger, B_y^\dagger] = \mathbf{0} \end{aligned} \tag{4}$$

(see [1, Lemma 2.1]).

Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$. For each $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we introduce

$$B(\varphi) := \int_{\mathbb{R}^d} \varphi(x) B_x\, dx, \quad B^\dagger(\varphi) := \int_{\mathbb{R}^d} \varphi(x) B_x^\dagger\, dx, \quad N(\varphi) := \int_{\mathbb{R}^d} \varphi(x) N_x\, dx. \tag{5}$$

By (4),

$$\begin{aligned} [B(\varphi), B^\dagger(\psi)] &= 2c\langle \varphi, \psi\rangle \mathbf{1} + 4N(\varphi\psi), \\ [N(\varphi), B^\dagger(\psi)] &= 2B^\dagger(\varphi\psi), \\ [N(\varphi), B(\psi)] &= -2B(\varphi\psi), \\ [N(\varphi), N(\psi)] = [B(\varphi), B(\psi)] &= [B^\dagger(\varphi), B^\dagger(\psi)] = \mathbf{0}, \qquad \phi, \psi \in \mathcal{S}(\mathbb{R}^d). \end{aligned} \tag{6}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\mathbb{R}^d, dx)$. The Lie algebra with generators $B(\varphi), B^\dagger(\varphi), N(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, and a central element $\mathbf{1}$ with relations (6) is called the square of white noise (SWN) algebra.

Now, one is interested in a Hilbert space representation of the SWN algebra with a cyclic vector Φ satisfying $B(\varphi)\Phi = 0$ (which is called a Fock representation). In [5], it was shown that a Fock representation of the SWN algebra exists if and only if the constant c is strictly positive. In what follows, we will suppose, for simplicity of notations that $c = 2$.

Let us now recall the Fock representation of the SWN algebra constructed in [3] (see also references therein).

For a real separable Hilbert space $\mathcal{H}$, denote by $\mathcal{F}(\mathcal{H})$ the symmetric Fock space over $\mathcal{H}$:

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\hat{\otimes} n} n!,$$

where $\hat{\otimes}$ stands for the symmetric tensor product. Thus, each $f \in \mathcal{F}(\mathcal{H})$ is of the form $f = (f^{(n)})_{n=0}^\infty$, where $f^{(n)} \in \mathcal{H}^{\hat{\otimes} n}$ and $\|f\|^2_{\mathcal{F}(\mathcal{H})} = \sum_{n=0}^\infty \|f^{(n)}\|^2_{\mathcal{H}^{\hat{\otimes} n}} n!$. Now take

$\mathcal{H}$ to be $L^2(\mathbb{R}^d, dx) \otimes \ell_2$, where the ℓ_2 space has the orthonormal basis $(e_n)_{n=1}^{\infty}$, $e_n = (0, \dots, 0, \underbrace{1}_{n\text{th place}}, 0, \dots)$.

Denote by $\mathfrak{F}$ the linear subspace of $\mathcal{F}(L^2(\mathbb{R}^d, dx) \otimes \ell_2)$ that is the linear span of the vacuum vector $\Omega = (1, 0, 0, \dots)$ and vectors of the form $(\varphi \otimes \xi)^{\otimes n}$, where $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $\xi \in \ell_{2,0}$, $n \in \mathbb{N}$. Here, $\ell_{2,0}$ denotes the linear subspace of ℓ_2 consisting of finite vectors, i.e., vectors of the form $\xi = (\xi_1, \xi_2, \dots, \xi_m, 0, 0, \dots)$, $m \in \mathbb{N}$. The set $\mathfrak{F}$ is evidently a dense subset of $\mathcal{F}(L^2(\mathbb{R}^d, dx) \otimes \ell_2)$.

Denote by J^+, J^0, J^- the linear operators in ℓ_2 with domain $\ell_{2,0}$ defined by the following formulas:

$$\begin{aligned} J^+ e_n &= \sqrt{n(n+1)}\, e_{n+1}, \\ J^0 e_n &= n e_n, \\ J^- e_n &= \sqrt{(n-1)n}\, e_{n-1}, \qquad n \in \mathbb{N}. \end{aligned} \tag{7}$$

Now, for each $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$ and $\xi \in \ell_{2,0}$, we set

$$\begin{aligned} B^\dagger(\varphi)(\psi \otimes \xi)^{\otimes n} &= 2(\varphi \otimes e_1) \hat{\otimes} (\psi \otimes \xi)^{\otimes n} + 2n((\varphi\psi) \otimes (J^+\xi))^{\otimes n}, \\ N(\varphi)(\psi \otimes \xi)^{\otimes n} &= 2n((\varphi\psi) \otimes J^0\xi)^{\otimes n}, \\ B(\varphi)(\psi \otimes \xi)^{\otimes n} &= 2n\langle \varphi, \psi\rangle \xi_1 (\psi \otimes \xi)^{\otimes(n-1)} + 2n((\varphi\psi) \otimes (J^-\xi))^{\otimes n}, \end{aligned} \tag{8}$$

where $n \in \mathbb{N}$, and $(\psi \otimes \xi)^{\otimes 0} := \Omega$. Thus,

$$\begin{aligned} B^\dagger(\varphi) &= 2A^+(\varphi \otimes e_1) + 2A^0(\varphi \otimes J^+), \\ N(\varphi) &= 2A^0(\varphi \otimes J^0), \\ B(\varphi) &= 2A^-(\varphi \otimes e_1) + 2A^0(\varphi \otimes J^-), \end{aligned} \tag{9}$$

where $A^+(\cdot)$, $A^0(\cdot)$, and $A^-(\cdot)$ are the creation, neutral, and annihilation operators in $\mathcal{F}(L^2(\mathbb{R}^d, dx) \otimes \ell_2)$, respectively. The operator $B^\dagger(\varphi)$ is the restriction of the adjoint operator of $B(\varphi)$ to $\mathfrak{F}$, while the operator $N(\varphi)$ is Hermitian. It is easy to see that the operators $B^\dagger(\varphi), N(\varphi), B(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, constitute a representation of the SWN algebra.

In what follows, the closure of a closable operator A will be denoted by $\widetilde{A}$. Since the adjoint operators of $B^\dagger(\varphi)$, $N(\varphi)$, $B(\varphi)$ are densely defined, they are closable.

The last part of [3] is devoted to studying those classical infinitely divisible processes which are built from the SWN in a similar way as the Wiener and Poisson processes are built from the usual white noise. So, for each parameter $\beta \geq 0$, we define

$$X_\beta(x) := B_x^\dagger + B_x + \beta N_x, \qquad x \in \mathbb{R}^d. \tag{10}$$

Notice that we want a formally self-adjoint process, so the parameter β must be real (we also exclude from consideration the case $\beta < 0$, since it may be treated by a trivial transformation of the case $\beta > 0$).

In view of (1) and (2), the only privileged parameter is $\beta = 2$, when $X_\beta(x)$ becomes the renormalized square of the classical white noise $b^\dagger(x) + b(x)$, see [1, Section 3].

Analogously to (5), we introduce, for each $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$X_\beta(\varphi) := \int_{\mathbb{R}^d} \varphi(x) X_\beta(x)\, dx = B^\dagger(\varphi) + B(\varphi) + \beta N(\varphi). \tag{11}$$

As easily seen, $\widetilde{X}_\beta(\varphi)$ is a self-adjoint operator.

In the case $d = 1$, it was shown in [3] that the quantum process $(\widetilde{X}_\beta(\chi_{[0,t]}))_{t\ge 0}$ (χ_Δ denoting the indicator function of a set Δ) is associated with a classical Lévy process $(Y_\beta(t))_{t\ge 0}$, which is a gamma process for $\beta = 2$, a Pascal process for $\beta > 2$, and a Meixner process for $0 \le \beta < 2$. (One has, of course, to extend the SWN algebra in order to include the operators indexed by the indicator functions, for example, to take the set $L^2(\mathbb{R}, dx) \cap L^\infty(\mathbb{R}, dx)$ instead of $\mathcal{S}(\mathbb{R})$.)

We also refer to [1, 2, 3] and references therein for a discussion of other aspects of the SWN.

On the other hand, in papers [16, 19, 20, 11] (see also [17, 12, 10, 13]), the Jacobi field of the Lévy processes of Meixner's type, i.e., the gamma, Pascal, and Meixner processes, was studied. Let us shortly explain this approach.

Let $\mathcal{S}'(\mathbb{R}^d)$ be the Schwartz space of tempered distributions. The $\mathcal{S}'(\mathbb{R}^d)$ is the dual space of $\mathcal{S}(\mathbb{R}^d)$ and the dualization between $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$ is given by the scalar product in $L^2(\mathbb{R}^d, dx)$. We will preserve the symbol $\langle\cdot,\cdot\rangle$ for this dualization. Let $\mathcal{C}(\mathcal{S}'(\mathbb{R}^d))$ denote the cylinder σ-algebra on $\mathcal{S}'(\mathbb{R}^d)$.

For each $\beta \ge 0$, we define a probability measure μ_β on $(\mathcal{S}'(\mathbb{R}^d)), \mathcal{C}(\mathcal{S}'(\mathbb{R}^d))$ by its Fourier transform

$$\int_{\mathcal{S}'(\mathbb{R}^d)} e^{i\langle\omega,\varphi\rangle}\, \mu_\beta(d\omega) = \exp\left[\int_{\mathbb{R}\times\mathbb{R}^d} (e^{is\varphi(x)} - 1 - is\varphi(x))\, \nu_\beta(ds)\, dx\right], \qquad \varphi \in \mathcal{S}(\mathbb{R}^d), \tag{12}$$

where the measure ν_β on $\mathbb{R}$ is specified as follows.

Let $\tilde\nu_\beta$ denote the probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ whose orthogonal polynomials $(\widetilde{P}_{\beta,n})_{n=0}^\infty$ with leading coefficient 1 satisfy the recurrence relation

$$s\widetilde{P}_{\beta,n}(s) = \widetilde{P}_{\beta,n+1}(s) + \beta(n+1)\widetilde{P}_{\beta,n}(s) + n(n+1)\widetilde{P}_{\beta,n-1}(s), \qquad n \in \mathbb{Z}_+,\ \widetilde{P}_{\beta,-1}(s) := 0. \tag{13}$$

By [14, Ch. VI, sect. 3], $(\widetilde{P}_{\beta,n})_{n=0}^\infty$ is a system of polynomials of Meixner's type, the measure $\widetilde{\nu}_\beta$ is uniquely determined by the above condition and is given as follows. For $\beta \in [0, 2)$,

$$\tilde\nu_\beta(ds) = \frac{\sqrt{4-\beta^2}}{2\pi} \left|\Gamma\big(1+i(4-\beta^2)^{-1/2}s\big)\right|^2 \exp\left[-s2(4-\beta^2)^{-1/2}\arctan\big(\beta(4-\beta^2)^{-1/2}\big)\right] ds$$

($\tilde{\nu}_\beta$ is a Meixner distribution), for $\beta = 2$

$$\tilde{\nu}_2(ds) = \chi_{(0,\infty)}(s)e^{-s}s\,ds$$

($\tilde{\nu}_2$ is a gamma distribution), and for $\beta > 2$

$$\tilde{\nu}_\beta(ds) = (\beta^2 - 4)\sum_{k=1}^{\infty} p_\beta^k\, k\, \delta_{\sqrt{\beta^2-4}\,k}, \qquad p_\beta := \frac{\beta - \sqrt{\beta^2-4}}{\beta + \sqrt{\beta^2-4}}$$

($\tilde{\nu}_\beta$ is now a Pascal distribution).

Notice that, for each $\beta \geq 0$, $\tilde{\nu}(\{0\}) = 0$, and hence, we may define

$$\nu_\beta(ds) := \frac{1}{s^2}\,\tilde{\nu}_\beta(ds). \tag{14}$$

Then, μ_β is the measure of gamma noise for $\beta = 2$, Pascal noise for $\beta > 2$, and Meixner noise for $\beta \in [0, 2)$. Indeed, for each $\beta \geq 0$, μ_β is a generalized process on $\mathcal{S}'(\mathbb{R}^d)$ with independent values (cf. [15]). Next, for each $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\int_{\mathcal{S}'(\mathbb{R}^d)} \langle \omega, \varphi\rangle^2\, \mu_\beta(d\omega) = \int_{\mathbb{R}^d} \varphi(x)^2\, dx. \tag{15}$$

Hence, for each $f \in L^2(\mathbb{R}^d, dx)$, we may define, in a standard way, the random variable $\langle \cdot, f\rangle$ from $L^2(\mathcal{S}'(\mathbb{R}^d), d\mu_\beta)$ satisfying (15) with $\varphi = f$.

Then, for each open, bounded set $\Delta \subset \mathbb{R}^d$, the distribution $\mu_{\beta,\Delta}$ of the random variable $\langle \cdot, \chi_\Delta\rangle$ under μ_β is given as follows. For $\beta > 2$, $\mu_{\beta,\Delta}$ is the negative binomial (Pascal) distribution

$$\mu_{\beta,\Delta} = (1 - p_\beta)^{|\Delta|} \sum_{k=0}^{\infty} \frac{\big(|\Delta|\big)_k}{k!}\, p_\beta^k\, \delta_{\sqrt{\beta^2-4}\,k - 2|\Delta|/(\beta + \sqrt{\beta^2-4})},$$

where for $r > 0$ $(r)_0 := 1$, $(r)_k := r(r+1)\cdots(r+k-1)$, $k \in \mathbb{N}$. For $\beta = 2$, $\mu_{2,\Delta}$ is the Gamma distribution

$$\mu_{2,\Delta}(ds) = \frac{(s + |\Delta|)^{|\Delta|-1} e^{-(s+|\Delta|)}}{\Gamma(|\Delta|)}\, \chi_{(0,\infty)}(s + |\Delta|)\, ds.$$

Finally, for $\beta \in [0, 2)$,

$$\begin{aligned}\mu_{\beta,\Delta}(ds) = \frac{(4-\beta^2)^{(|\Delta|-1)/2}}{2\pi\Gamma(|\Delta|)} &\big|\Gamma\big(|\Delta|/2 + i(4-\beta^2)^{-1/2}(s + \beta|\Delta|/2)\big|^2 \\ &\times \exp\big[-(2s + \beta|\Delta|)(4-\beta^2)^{-1/2}\arctan\big(\beta(4-\beta^2)^{-1/2}\big)\big]\, ds.\end{aligned}$$

Here, $|\Delta| := \int_\Delta dx$.

We denote by $\mathcal{P}(\mathcal{S}'(\mathbb{R}^d))$ the set of continuous polynomials on $\mathcal{S}'(\mathbb{R}^d)$, i.e., functions on $\mathcal{S}'(\mathbb{R}^d)$ of the form

$$F(\omega)=\sum_{i=0}^{n}\langle\omega^{\otimes i},f^{(i)}\rangle,\qquad \omega^{\otimes 0}{:=}1,\ f^{(i)}\in\mathcal{S}(\mathbb{R}^d)^{\hat{\otimes} i},\ i=0,\dots,n,\ n\in\mathbb{Z}_+.$$

The greatest number i for which $f^{(i)}\neq 0$ is called the power of a polynomial. We denote by $\mathcal{P}_n(\mathcal{S}'(\mathbb{R}^d))$ the set of continuous polynomials of power $\le n$.

The set $\mathcal{P}(\mathcal{S}'(\mathbb{R}^d))$ is a dense subset of $L^2(\mathcal{S}'(\mathbb{R}^d),d\mu_\beta)$. Let $\mathcal{P}_n^{\sim}(\mathcal{S}'(\mathbb{R}^d))$ denote the closure of $\mathcal{P}_n(\mathcal{S}'(\mathbb{R}^d))$ in $L^2(\mathcal{S}'(\mathbb{R}^d),d\mu_\beta)$, let $\mathbf{P}_n(\mathcal{S}'(\mathbb{R}^d))$, $n\in\mathbb{N}$, denote the orthogonal difference $\mathcal{P}_n^{\sim}(\mathcal{S}'(\mathbb{R}^d))\ominus\mathcal{P}_{n-1}^{\sim}(\mathcal{S}'(\mathbb{R}^d))$, and let $\mathbf{P}_0(\mathcal{S}'(\mathbb{R}^d)){:=}\mathcal{P}_0^{\sim}(\mathcal{S}'(\mathbb{R}^d))$. We evidently have the orthogonal decomposition

$$L^2(\mathcal{S}'(\mathbb{R}^d),d\mu_\beta)=\bigoplus_{n=0}^{\infty}\mathbf{P}_n(\mathcal{S}'(\mathbb{R}^d)).\tag{16}$$

For a monomial $\langle\omega^{\otimes n},f^{(n)}\rangle$, $f^{(n)}\in\mathcal{S}(\mathbb{R}^d)^{\hat{\otimes} n}$, we denote by ${:}\langle\omega^{\otimes n},f^{(n)}\rangle{:}$ the orthogonal projection of $\langle\omega^{\otimes n},f^{(n)}\rangle$ onto $\mathbf{P}_n(\mathcal{S}'(\mathbb{R}^d))$. The set $\{{:}\langle\omega^{\otimes n},f^{(n)}\rangle{:},\ f^{(n)}\in\mathcal{S}(\mathbb{R}^d)^{\hat{\otimes} n}\}$ is dense in $\mathbf{P}_n(\mathcal{S}'(\mathbb{R}^d))$.

Denote by $\mathbb{Z}^\infty_{+,0}$ the set of all sequences α of the form $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n,0,0,\dots)$, $\alpha_i\in\mathbb{Z}_+$, $n\in\mathbb{N}$. Let $|\alpha|{:=}\sum_{i=1}^\infty\alpha_i$, evidently $|\alpha|\in\mathbb{Z}_+$. For each $\alpha\in\mathbb{Z}^\infty_{+,0}$, $1\alpha_1+2\alpha_2+\dots=n$, $n\in\mathbb{N}$, and for any function $f^{(n)}:(\mathbb{R}^d)^n\to\mathbb{R}$ we define a function $D_\alpha f^{(n)}:(\mathbb{R}^d)^{|\alpha|}\to\mathbb{R}$ by setting

$$\begin{aligned}(D_\alpha f^{(n)})(x_1,\dots,x_{|\alpha|}){:=}f^{(n)}(x_1,\dots,x_{\alpha_1},\underbrace{x_{\alpha_1+1},x_{\alpha_1+1}}_{\text{2 times}},\underbrace{x_{\alpha_1+2},x_{\alpha_1+2}}_{\text{2 times}},\dots,\underbrace{x_{\alpha_1+\alpha_2},x_{\alpha_1+\alpha_2}}_{\text{2 times}},\\ \underbrace{x_{\alpha_1+\alpha_2+1},x_{\alpha_1+\alpha_2+1},,x_{\alpha_1+\alpha_2+1}}_{\text{3 times}},\dots).\end{aligned}$$

We define a scalar product on $\mathcal{S}(\mathbb{R}^d)^{\hat{\otimes} n}$ by setting for any $f^{(n)},g^{(n)}\in\mathcal{S}(\mathbb{R}^d)^{\hat{\otimes} n}$

$$\begin{aligned}(f^{(n)},g^{(n)})_{\mathcal{F}^{(n)}_{\mathrm{Ext}}(L^2(\mathbb{R}^d,dx))}{:=}\sum_{\alpha\in\mathbb{Z}^\infty_{+,0}:\,1\alpha_1+2\alpha_2+\dots=n}K_\alpha\int_{X^{|\alpha|}}(D_\alpha f^{(n)})(x_1,\dots,x_{|\alpha|})\\ \times(D_\alpha g^{(n)})(x_1,\dots,x_{|\alpha|})\,dx_1\cdots dx_{|\alpha|},\end{aligned}\tag{17}$$

where

$$K_\alpha=\frac{n!}{\alpha_1!\,1^{\alpha_1}\alpha_2!\,2^{\alpha_2}\cdots}\,.\tag{18}$$

Let $\mathcal{F}^{(n)}_{\mathrm{Ext}}(L^2(\mathbb{R}^d,dx))$ be the closure of $\mathcal{S}(\mathbb{R}^d)^{\hat{\otimes} n}$ in the norm generated by (17), (18). The extended Fock space $\mathcal{F}_{\mathrm{Ext}}(L^2(\mathbb{R}^d,dx))$ over $L^2(\mathbb{R}^d,dx)$ is defined as

$$\mathcal{F}_{\mathrm{Ext}}(L^2(\mathbb{R}^d,dx)){:=}\bigoplus_{n=0}^{\infty}\mathcal{F}^{(n)}_{\mathrm{Ext}}(L^2(\mathbb{R}^d,dx))\,n!,\tag{19}$$

where $\mathcal{F}^{(n)}_{\mathrm{Ext}}(L^2(\mathbb{R}^d,dx)):=\mathbb{R}$. We also denote by Ω the vacuum vector in $\mathcal{F}_{\mathrm{Ext}}(L^2(\mathbb{R}^d,dx))$: $\Omega=(1,0,0,\dots)$.

For any $f^{(n)},g^{(n)}\in\mathcal{S}(\mathbb{R}^d)^{\hat{\otimes}n}$, $n\in\mathbb{N}$, we have

$$\int_{\mathcal{S}'(\mathbb{R}^d)} {:}\langle\omega^{\otimes n},f^{(n)}\rangle{:}\,{:}\langle\omega^{\otimes n},g^{(n)}\rangle{:}\,\mu_\beta(d\omega)=(f^{(n)},g^{(n)})_{\mathcal{F}^{(n)}_{\mathrm{Ext}}(L^2(\mathbb{R}^d,dx))}\,n!\,. \tag{20}$$

Therefore, for each $f^{(n)}\in\mathcal{F}^{(n)}_{\mathrm{Ext}}(L^2(\mathbb{R}^d,dx))$, we can define, a random variable ${:}\langle\cdot^{\otimes n},f^{(n)}\rangle{:}$ from $L^2(\mathcal{S}'(\mathbb{R}^d),d\mu_\beta)$ such that equality (20) remains true for any $f^{(n)},g^{(n)}\in\mathcal{F}^{(n)}_{\mathrm{Ext}}(L^2(\mathbb{R}^d,dx))$, and furthermore

$$\begin{aligned}&\mathcal{F}_{\mathrm{Ext}}(L^2(\mathbb{R}^d,dx))\ni f=(f^{(n)})_{n=0}^\infty\mapsto\\&\qquad\mapsto U_\beta f=(U_\beta f)(\omega)=\sum_{n=0}^\infty {:}\langle\omega^{\otimes n},f^{(n)}\rangle{:}\in L^2(\mathcal{S}'(\mathbb{R}^d),d\mu_\beta)\end{aligned} \tag{21}$$

is unitary.

We denote by $\mathcal{F}_{\mathrm{fin}}(\mathcal{S}(\mathbb{R}^d))$ the dense subset of $\mathcal{F}_{\mathrm{Ext}}(L^2(\mathbb{R}^d,dx))$ consisting of vectors of the form $(f^{(0)},f^{(1)},\dots,f^{(n)},0,0,\dots)$, where $f^{(i)}\in\mathcal{S}(\mathbb{R}^d)^{\hat{\otimes}i}$. For each $\beta\ge 0$ and each $\varphi\in\mathcal{S}(\mathbb{R}^d)$, we define an operator $a_\beta(\varphi)$ on $\mathcal{F}_{\mathrm{fin}}(\mathcal{S}(\mathbb{R}^d))$ by the following formula:

$$a_\beta(\varphi)=a^+(\varphi)+\beta a^0(\varphi)+a^-(\varphi).$$

Here, $a^+(\xi)$ is the standard creation operator:

$$a^+(\varphi)f^{(n)}:=\varphi\hat{\otimes}f_n,\qquad f^{(n)}\in\mathcal{S}(\mathbb{R}^d)^{\hat{\otimes}n},\ n\in\mathbb{Z}_+, \tag{22}$$

$a^0(\varphi)$ is the standard neutral operator:

$$(a^0(\varphi)f^{(n)})(x_1,\dots,x_n)=\big(\varphi(x_1)+\dots+\varphi(x_n)\big)f_n(x_1,\dots,x_n), \tag{23}$$

and

$$a^-(\varphi)=a_1^-(\varphi)+a_2^-(\varphi), \tag{24}$$

where $a_1^-(\varphi)$ is the standard annihilation operator:

$$(a_1^-(\varphi)f^{(n)})(x_1,\dots,x_{n-1})=n\int_{\mathbb{R}^d}\varphi(x)f^{(n)}(x,x_1,\dots,x_{n-1})\,dx, \tag{25}$$

and

$$(a_2^-(\varphi)f^{(n)})(x_1,\dots,x_{n-1})=n(n-1)(\varphi(x_1)f^{(n)}(x_1,x_1,x_2,x_3,\dots,x_{n-1}))^\sim, \tag{26}$$

$(\cdot)^\sim$ denoting symmetrization of a function.

Denote by $\partial_x^\dagger$, ∂_x the standard creation and annihilation operators at point $x \in \mathbb{R}^d$:

$$\partial_x^\dagger f^{(n)} = \delta_x \hat{\otimes} f^{(n)}, \quad \partial_x f^{(n)}(x_1, \dots, x_{n-1}) = n f^{(n)}(x, x_1, \dots, x_{n-1}).$$

Then, at least formally, we have the following representation:

$$a^+(\varphi) = \int_{\mathbb{R}^d} \varphi(x) \partial_x^\dagger \, dx, \quad a^0(\varphi) = \int_{\mathbb{R}^d} \varphi(x) \partial_x^\dagger \partial_x \, dx, \quad a^-(\varphi) = \int_{\mathbb{R}^d} \varphi(x)(\partial_x + \partial_x^\dagger \partial_x^2) \, dx, \tag{27}$$

so that

$$a_\beta(\varphi) = \int_{\mathbb{R}^d} \varphi(x)(\partial_x^\dagger + \beta \partial_x^\dagger \partial_x + \partial_x + \partial_x^\dagger \partial_x^2) \, dx. \tag{28}$$

(In fact, equalities (27), (28) may be given a precise meaning, cf. [16, 19].)

The operators $a_\beta(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, are essentially self-adjoint on $\mathcal{F}_{\mathrm{fin}}(\mathcal{S}(\mathbb{R}^d))$ and the image of any $\tilde{a}_\beta(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, under the unitary U_β is the operator of multiplication by the random variable $\langle \cdot, \varphi \rangle$. Thus, $(\tilde{a}(\varphi))_{\varphi \in \mathcal{S}(\mathbb{R}^d)}$ is the Jacobi field of μ_β, see [8, 9, 18, 11] and the references therein.

The functional realization of the operators $a^+(\varphi)$, $a^0(\varphi)$, $a^-(\varphi)$, i.e., the explicit action of the the image of these operators under the unitary U_β is discussed in [16, 19].

A direct computation shows that the operators $2a^+(\varphi), 2a^0(\varphi), 2a^-(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, satisfy the commutation relations (6), and hence generate a SWN algebra. In fact, we have the following result:

Theorem 1 *For each $\beta \geq 0$, there exists a unitary operator*

$$I_\beta : \mathcal{F}(L^2(\mathbb{R}^d, dx) \otimes \ell_2) \to \mathcal{F}_{\mathrm{Ext}}(L^2(\mathbb{R}^d, dx))$$

such that $I_\beta \Omega = \Omega$ and the operators $\widetilde{X}_\beta(\varphi)$, $\widetilde{B}^\dagger(\varphi)$, $\widetilde{N}(\varphi)$, $\widetilde{B}(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, are unitarily isomorphic under I_β to two times the operators $\tilde{a}(\varphi)$, $\tilde{a}^+(\varphi)$, $\tilde{a}^0(\varphi)$, $\tilde{a}^-(\varphi)$, respectivlely.

Notice that the unitary operator

$$\mathcal{U}_\beta := U_\beta I_\beta : \mathcal{F}(L^2(\mathbb{R}^d, dx) \otimes \ell_2) \to L^2(\mathcal{S}'(\mathbb{R}^d), d\mu_\beta)$$

has the following properties: $\mathcal{U}_\beta \Omega = 1$ and

$$\mathcal{U}_\beta \widetilde{X}_\beta(\varphi) \mathcal{U}_\beta^{-1} = 2\langle \cdot, \varphi \rangle \cdot, \qquad \varphi \in \mathcal{S}(\mathbb{R}^d)$$

(compare with [3])

By virtue of (5), (10), (27), and (28), we get from Theorem 1:

$$B_x = 2(\partial_x + \partial_x^\dagger \partial_x^2), \quad N_x = 2\partial_x^\dagger \partial_x, \quad B_x^\dagger = 2\partial_x^\dagger, \tag{29}$$

and

$$X_\beta(x) = 2(\partial_x^\dagger + \beta \partial_x^\dagger \partial_x + \partial_x + \partial_x^\dagger \partial_x^2), \qquad x \in \mathbb{R}^d$$

(where the equalities are to be understood in the sense of the unitary isomorphism). The reader is advised to compare (29) with the informal representation (2).

2 Proof of the theorem

The proof of Theorem 1 is essentially based on the results of [20]. By (9) and (11), we get, for each $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$X_\beta(\varphi) = 2(A^+(\varphi \otimes e_1) + A^0(\varphi \otimes J_\beta) + A^-(\varphi \otimes e_1)),$$

where

$$J_\beta := J^+ + \beta J^0 + J^-.$$

By (7), the operator J_β is given by a Jacobi matrix (see e.g. [7]). Furthermore, J_β is essentially self-adjoint on $\ell_{2,0}$ and, by (13), $\tilde{\nu}_\beta$ is the spectral measure of $\widetilde{J}_\beta$. The latter means that there exists a unitary operator

$$I_\beta^{(1)} : \ell_2 \to L^2(\mathbb{R}, d\tilde{\nu}_\beta)$$

such that $I_\beta^{(1)} e_1 = 1$ and, under $I_\beta^{(1)}$, the operator $\widetilde{J}_\beta$ goes over into the operator of multiplication by s.

Next, by (14), the operator

$$L^2(\mathbb{R}, d\tilde{\nu}_\beta) \ni f \mapsto I_\beta^{(2)} f = (I_\beta^{(2)} f)(s) := f(s)s \in L^2(\mathbb{R}, d\nu_\beta)$$

is unitary. Setting

$$I_\beta^{(3)} := I_\beta^{(2)} I_\beta^{(1)} : \ell_2 \to L^2(\mathbb{R}, d\nu_\beta),$$

we get a unitary operator such that $I_\beta^{(3)} e_1 = (I_\beta^{(3)} e_1)(s) = s$ and, under $I_\beta^{(3)}$, $\widetilde{J}_\beta$ goes over into the operator of multiplication by s.

Using $I_\beta^{(3)}$, we can naturally construct a unitary operator

$$I_\beta^{(4)} : \mathcal{F}(L^2(\mathbb{R}^d, dx) \otimes \ell_2) \to \mathcal{F}(L^2(\mathbb{R}^d, dx) \otimes L^2(\mathbb{R}, d\nu_\beta))$$

such that $I_\beta^{(4)} \Omega = \Omega$ and, under $I_\beta^{(4)}$, the operator $X_\beta(\varphi)$ goes over into the operator

$$\mathcal{X}_\beta(\varphi) = 2(A^+(\varphi \otimes s) + A^0(\varphi \otimes s) + A^-(\varphi \otimes s)).$$

It follows from [20] that there exists a unitary operator

$$I_\beta^{(5)} : \mathcal{F}(L^2(\mathbb{R}^d, dx) \otimes L^2(\mathbb{R}, d\nu_\beta)) \to L^2(\mathcal{S}'(\mathbb{R}^d), d\mu_\beta)$$

such that $I_\beta^{(5)} \Omega = 1$ and, under $I_\beta^{(5)}$, the operator $\widetilde{\mathcal{X}}_\beta(\varphi)$ goes over into the operator of multiplication by $2\langle \cdot, \varphi \rangle$.

We define the unitary

$$I_\beta := U_\beta^{-1} I_\beta^{(5)} I_\beta^{(4)} : \mathcal{F}(L^2(\mathbb{R}^d, dx) \otimes \ell_2) \to \mathcal{F}_{\mathrm{Ext}}(L^2(\mathbb{R}^d, dx)),$$

where U_β is given by (21). We evidently get $I_\beta\Omega = \Omega$ and $\tilde{a}(\varphi) = I_\beta^{-1}\widetilde{X}_\beta(\varphi)I_\beta^{-1}$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Next, we denote by $\mathfrak{G}$ the subset of $\mathcal{F}_{\mathrm{Ext}}(L^2(\mathbb{R}^d, dx))$ defined as the linear span of Ω and the vectors of the form $\varphi^{\otimes n}$, where $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $n \in \mathbb{N}$. We note:

$$(I_\beta^{(3)} e_n)(s) = P_{\beta,n}(s), \qquad n \in \mathbb{N},$$

where

$$P_{\beta,n}(s) := s\widetilde{P}_{\beta,n-1}(s), \qquad n \in \mathbb{N},$$

and $(\widetilde{P}_{\beta,n})_{n=0}^{\infty}$ are defined by (13). Hence, by [20, Sect. 4 and Corollary 5.1],

$$\mathfrak{G} \subset I_\beta \mathfrak{F}.$$

Furthermore, by (7), (8), (22)–(26) and by [20, Corollary 5.1], we get:

$$\begin{aligned} I_\beta B^\dagger(\varphi) I_\beta^{-1} \restriction \mathfrak{G} &= a^+(\varphi) \restriction \mathfrak{G}, \\ I_\beta N(\varphi) I_\beta^{-1} \restriction \mathfrak{G} &= a^0(\varphi) \restriction \mathfrak{G}, \\ I_\beta B(\varphi) I_\beta^{-1} \restriction \mathfrak{G} &= a^-(\varphi) \restriction \mathfrak{G}, \qquad \varphi \in \mathcal{S}(\mathbb{R}^d). \end{aligned} \tag{30}$$

We now endow $\mathcal{F}_{\mathrm{fin}}(\mathcal{S}(\mathbb{R}^d))$ with the topology of the topological direct sum of the spaces $\mathcal{F}_n(\mathcal{S}(\mathbb{R}^d))$. Thus, the convergence in $\mathcal{F}_{\mathrm{fin}}(\mathcal{S}(\mathbb{R}^d))$ means the uniform finiteness and the coordinate-wise convergence in each $\mathcal{F}_n(\mathcal{S}(\mathbb{R}^d))$. As easily seen, $\mathfrak{G}$ is a dense subset of $\mathcal{F}_{\mathrm{fin}}(\mathcal{S}(\mathbb{R}^d))$. Since the operators $a^+(\varphi)$, $a^0(\varphi)$, and $a^-(\varphi)$ act continuously on $\mathcal{F}_{\mathrm{fin}}(\mathcal{S}(\mathbb{R}^d))$ and since $\mathcal{F}_{\mathrm{fin}}(\mathcal{S}(\mathbb{R}^d))$ is continuously embedded into $\mathcal{F}_{\mathrm{Ext}}(L^2(\mathbb{R}^d, dx))$ (cf. [16, p. 37]), the closure of the operators $a^+(\varphi)$, $a^0(\varphi)$, and $a^-(\varphi)$ restricted to $\mathfrak{G}$ coincides with $\tilde{a}^+(\varphi)$, $\tilde{a}^0(\varphi)$, and $\tilde{a}^-(\varphi)$, respectively. Hence, by (30), $\widetilde{B}^\dagger(\varphi)$, $\widetilde{N}(\varphi)$, and $\widetilde{N}(\varphi)$ are extensions of the operators $\tilde{a}^+(\varphi)$, $\tilde{a}^0(\varphi)$, and $\tilde{a}^-(\varphi)$, respectively.

Finally, analogously to the proof of [20, Theorem 6.1], we conclude that $I_\beta\mathfrak{F}$ is a subset of the domain of $\tilde{a}^+(\varphi)$, respectively $\tilde{a}^0(\varphi)$, respectively $\tilde{a}^-(\varphi)$, and furthermore

$$\begin{aligned} I_\beta B^\dagger(\varphi) I_\beta^{-1} &= \tilde{a}^+(\varphi) \restriction I_\beta\mathfrak{F}, \\ I_\beta N(\varphi) I_\beta^{-1} &= \tilde{a}^0(\varphi) \restriction I_\beta\mathfrak{F}, \\ I_\beta B(\varphi) I_\beta^{-1} &= \tilde{a}^-(\varphi) \restriction I_\beta\mathfrak{F}, \qquad \varphi \in \mathcal{S}(\mathbb{R}^d). \end{aligned}$$

This yields:

$$\begin{aligned} I_\beta \widetilde{B}^\dagger(\varphi) I_\beta^{-1} &= \tilde{a}^+(\varphi), \\ I_\beta \widetilde{N}(\varphi) I_\beta^{-1} &= \tilde{a}^0(\varphi), \\ I_\beta \widetilde{B}(\varphi) I_\beta^{-1} &= \tilde{a}^-(\varphi), \qquad \varphi \in \mathcal{S}(\mathbb{R}^d), \end{aligned}$$

which concludes the proof.

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